About dual cube theorems

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Abstract

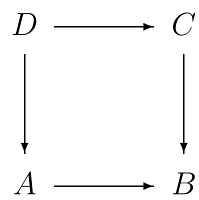
Most of the properties of the category of Lusternik-Schnirelmann come from the cube theorems of Mather, especially the second. The duals of these theorems are false, which makes the dual category problematic. However, a weaker version of the second cube theorem is sufficient to get the usual properties of the LS-category. In this note, we show that this weaker version of the dual of the first theorem is false. The weaker version of the dual of the second cube theorem remains an open problem.

We place ourselves in a category where 'homotopy commutative diagrams' has a meaning. The category of topological spaces is the first example; see [8]. The full subcategory $\mathbf{C}_{cf}$ of cofibrant and fibrant objects of any model category $\mathbf{C}$ is also suitable; here 'homotopy' is to be understood in the sense of Quillen [9]. Such categories (when pointed) have been shown to be the right place to build the theory of LS-category of their objects (spaces, simplicial sets, differential graded algebras, differential graded Lie algebras, etc.), provided an additional axiom holds: Axiom 2 in this paper; see [1],[2], [3], [6]. In this paper we shall show that a weaker one: Axiom 8, is sufficient, and we shall also consider the dual of this axiom, which is related to the co-LS-category.

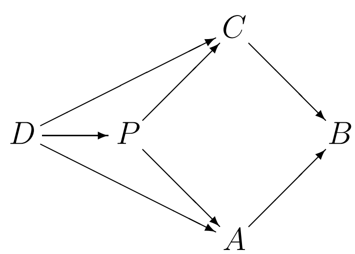
In such a category, a 'homotopy pull back' is a homotopy commutative square:

$$\begin{array}{ccc} P & \longrightarrow & C \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

such that for any homotopy commutative diagram

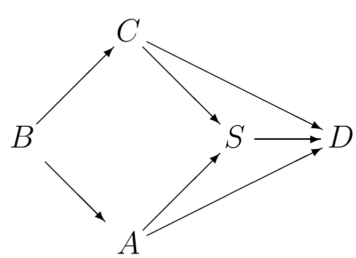


there exists a map $D \to P$ (here called the 'induced' map) and a homotopy commutative diagram (here called 'homotopy pull back diagram'):



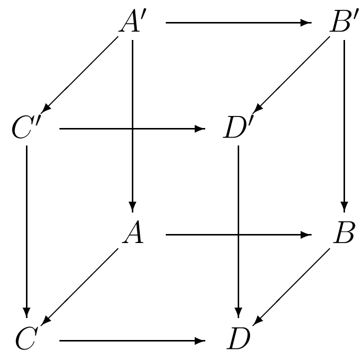
which is unique up to equivalences of homotopies.

The dual notion of 'homotopy pull back' is 'homotopy push out'. By 'dual', we mean 'Eckmann-Hilton duality', i.e. keep the same diagram but reverse all arrows. So a 'homotopy push out diagram' is a diagram like this:



where the square $B - C - S - A$ is a homotopy push out. If, moreover, the square $B - C - D - A$ is a homotopy pull back, we say that S is the 'join' of A and C over D and we write $S \simeq A \bowtie_D C$.

Axiom 1 *Consider a homotopy commutative cube*



where: (i) the left and rear faces are homotopy pull backs and (ii) the top and bottom faces are homotopy push outs. Then the front and right faces are homotopy pull backs.

M. Mather [8] has proved that this Axiom holds in the category of topological spaces. This is what he called the 'first cube theorem':

Axiom 2 (Cube axiom.) *If in a homotopy commutative cube as above, (i) all vertical faces are homotopy pull backs, and (ii) the bottom face is a homotopy push out, then the top face is a homotopy push out.*

Whenever the spaces are CW-complexes, it is easy to see that the first Axiom implies the second. But Mather [8] proved that the second Axiom holds in the whole category of topological spaces. This is what he called the 'second cube theorem'.

These theorems lead to important applications: [2], [4], [7], especially the theory of LS-category relies on these theorems.

The dual of Axiom 1 is:

Axiom 3 *If in a commutative cube as above, (i) the right and front faces are homotopy push outs and (ii) the top and bottom faces are homotopy pull backs, then the left and rear faces are homotopy push outs.*

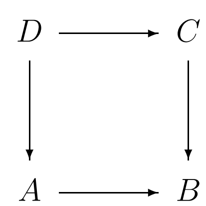
The dual of Axiom 2 is:

Axiom 4 *If in a homotopy commutative cube as above (i) all vertical faces are homotopy push outs, and (ii) the top face is a homotopy pull back, then the bottom face is a homotopy pull back.*

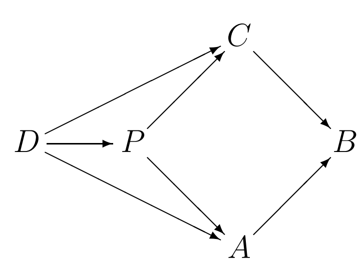
It is easy to find a counterexample of Axiom 4 in the category of topological spaces [2], making the notion of co-LS-category problematic.

However, the theory of LS-category can be developped with a weaker version of Axiom 2. We give this weaker version, Axiom 8, after the following definitions:

Definition 5 *Any homotopy commutative square*



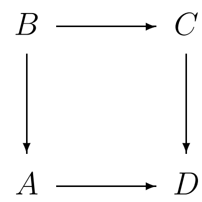
will be called 'homotopy pull back extension' (or 'hpbe' for short) if for one (equivalently: any) homotopy pull back diagram



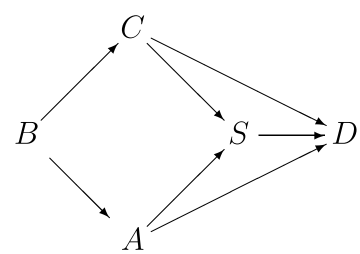
the induced map $D \to P$ *has a homotopy section (so* P *is a homotopy retract of* D*).*

Analogously:

Definition 6 *Any homotopy commutative square*



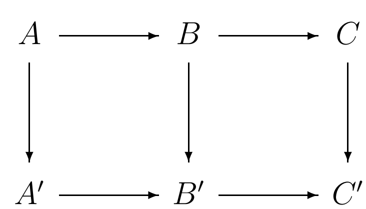
will be called 'homotopy push out extension' (or 'hpoe' for short) if for one (equivalently: any) homotopy push out diagram



the induced map $S \to D$ has a homotopy section (so D is a homotopy retract of S).

We have the following (easy) lemma:

Lemma 7 *If in any homotopy commutative diagram*



the right square is a homotopy pull back, then: the outer rectangle is a homotopy pull back extension if and only if the left square is a homotopy pull back extension.

To prove this lemma, just note that P is the homotopy pull back of

$A' \to B'$ and $B \to B'$ if and only if P is the homotopy pull back of $A' \to C'$ and $C \to C'$.

Axiom 8 *If in a homotopy commutative cube as above (i) all vertical faces are homotopy pull backs, and (ii) the bottom face is a homotopy push out, then the top face is a homotopy push out extension.*

If this Axiom holds then we have:

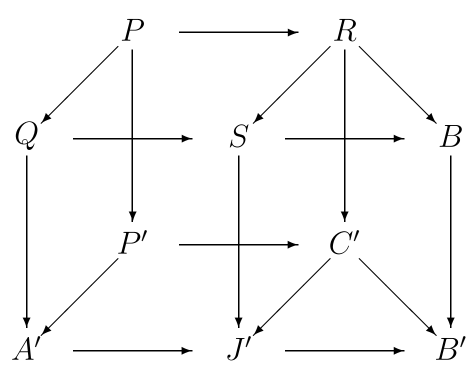
Theorem 9 (Join theorem.) (Compare [2], 3.1.) *Consider two homotopy pull backs extensions*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longleftarrow & C \\ \downarrow & hpbe & \downarrow & hpbe & \downarrow \\ A' & \longrightarrow & B' & \longleftarrow & C' \end{array}$$

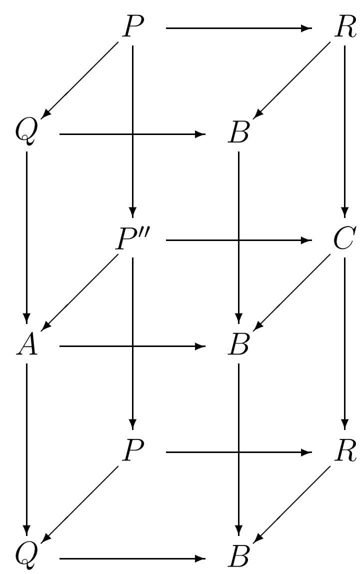
Then we have a homotopy pull back extension

$$\begin{array}{ccc} A \bowtie_B C & \longrightarrow & B \\ \downarrow & hpbe & \downarrow \\ A' \bowtie_{B'} C' & \longrightarrow & B' \end{array}$$

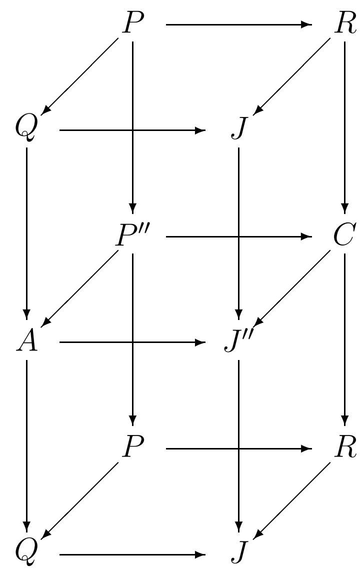
Proof. Consider the following construction:



where $P'-C'-B'-A'$ is a hpb, $P'-C'-J'-A'$ is a hpo, so $J' \simeq A' \bowtie_{B'} C'$, all vertical faces are hpb. Note that $P-R-S-Q$ is a hpoe. Since Q is a homotopy retract of A and R is a homotopy retract of C, we can construct the following homotopy commutative diagram:



where P'' is the hpb of $A \to B$ and $C \to B$ and the maps $P'' \to P$ and $P \to P''$ are induced by the hpb. The composite of these two maps is the identity up to homotopy. Now we can construct the following homotopy commutative diagram:



where the horizontal squares are hpo, so $J'' \simeq A \bowtie_B C$, and $J \to S$ has a homotopy section as $P - R - S - Q$ is a hpoe. Thus S is a homotopy retract of J and J is a homotopy retract of J'', so S is a homotopy retract of J''. $\square$

Now with this theorem, if the category is pointed by a zero object $*$, we can build the whole theory of LS-category. For instance consider the Ganea construction, i.e. maps $g_n : G_nB \to B$ such that

$$g_0 : G_0B \simeq * \to B, \qquad g_n : G_nB \simeq G_{n-1}B \bowtie_B * \to B$$

and the Whitehead construction, i.e. maps $t_n : T_nB \to B^{n+1}$ such that

$$t_0 : T_0B \simeq * \to B, \qquad t_n : T_nB \simeq B^n \bowtie_{B^{n+1}} (T_{n-1} \times B) \to B^{n+1}$$

We have:

Theorem 10 (Compare [2], 5.5.) *For all $n \geq 1$ there is a homotopy pull back extension*

$$\begin{array}{ccc} G_nB & \longrightarrow & T_nB \\ {\scriptstyle g_n}\downarrow & hpbe & \downarrow{\scriptstyle t_n} \\ B & \xrightarrow[\Delta_{n+1}]{} & B^{n+1} \end{array}$$

Proof. By induction on n. The case $n = 0$ is immediate as $g_0 = t_0$ and Δ_1 is the identity. Assume the result for $n = m - 1$.

Consider the following homotopy commutative diagram

$$\begin{array}{ccccc} G_{m-1}B & \longrightarrow & T_{m-1}B \times B & \longrightarrow & T_{m-1}B \\ \downarrow & & \downarrow & & \downarrow \\ B & \underset{\Delta_{m+1}}{\longrightarrow} & B^m \times B & \longrightarrow & B^m \end{array}$$

As the right square is a hpb and the outer rectangle is a hpbe, we obtain that the left square is a hpbe (†).

On the other hand, consider the following homotopy commutative diagram

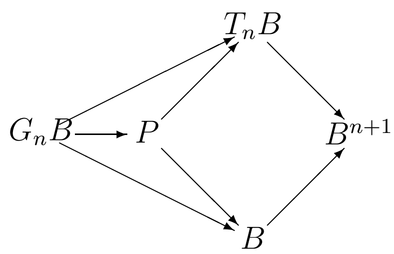
$$\begin{array}{ccccc} * & \longrightarrow & B^m & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ B & \underset{\Delta_{m+1}}{\longrightarrow} & B^m \times B & \longrightarrow & B \end{array}$$

As the right square and the outer rectangle are hpb, we obtain that the left square is a hpb (‡).

Now apply join theorem I to (†) and (‡) to get the result for $n = m$. $\square$

Corollary 11 *For all* $n \geq 1$, g_n *has a homotopy section if and only if* Δ_{n+1} *admits a homotopy factorization through* t_n.

Proof. Consider the following homotopy pull back diagram



We know by the last theorem that the induced map $G_nB \to P$ has a homotopy section s. If Δ_{n+1} has a lifting through t_n, we can consider the map $j : B \to P$ induced by the maps id_B and $B \to T_nB$. The composite sj is the desired section of g_n. □

To end this note, we give a result about the weaker version of the dual of the first cube theorem, here called Axiom 13.

Before we state this, we need to give the dual notion of homotopy pull back extension, which we will call 'homotopy push out coextension':

Definition 12 *Any homotopy commutative square as in Definition 6 will be called 'homotopy push out coextension' (or 'hpoc' for short) if for one (equivalently: any) homotopy push out diagram the induced map $S \to D$ has a homotopy retraction (so S is a homotopy retract of D).*

The weaker version of Axiom 3 is:

Axiom 13 *If in a homotopy commutative cube (i) the right and front faces are homotopy push outs, and (ii) the top and bottom faces are homotopy pull backs, then the left and rear faces are homotopy push out coextension.*

The weaker version of Axiom 4, also the dual of Axiom 8, is:

Axiom 14 *If in a homotopy commutative cube (i) all vertical faces are homotopy push outs, and (ii) the top face is a homotopy pull back, then the bottom face is a homotopy pull back coextension.*

Theorem 15 *Let the following diagram*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ X' & \xrightarrow[h']{} & Y' \end{array}$$

be a homotopy push out where h is a homotopy monomorphism. If Axiom 13 holds, then h' is a homotopy monomorphism, too.

Corollary 16 *Axiom 13 does not hold in the category of topological spaces.*

Indeed, in this category there exist homotopy push outs as above where h is a homotopy monomorphism and h' is not [5].

It is important to note that the fact whether Axiom 14 holds or not in the category of topological space remains an open problem. Dualizing Join theorem, we know that if this Axiom holds, then we can build the theory of co-LS-category.

Now, we give the proof of Theorem 15:

Proof.

Consider the two homotopy pull backs

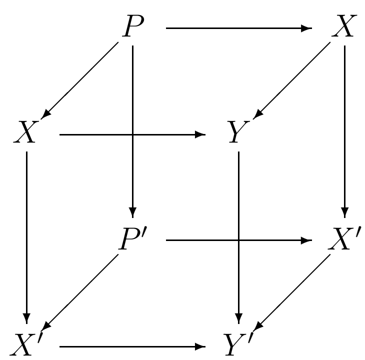
$$\begin{array}{ccc} P & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[h]{} & Y \end{array}$$

and

$$\begin{array}{ccc} P' & \xrightarrow{p'_1} & X' \\ {\scriptstyle p'_2}\downarrow & & \downarrow{\scriptstyle h'} \\ X' & \xrightarrow[h']{} & Y' \end{array}$$

Since h is a homotopy monomorphism, and $h \circ p_1$ and $h \circ p_2$ are homotopic, it follows that p_1 and p_2 are homotopic.

Consider the diagram



in which (i) the right and front faces are homotopy push out and (ii) the top and bottom faces are homotopy pull backs, and $\beta : P \to P'$ is the induced map by the pull back at the bottom.

Let S be the homotopy push out :

$$\begin{array}{ccc} P & \xrightarrow{p_1} & X \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle q_2} \\ P' & \xrightarrow[q_1]{} & S \end{array}$$

Since the square

$$\begin{array}{ccc} P & \xrightarrow{p_1} & X \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle t} \\ P' & \xrightarrow[p_1']{} & X' \end{array}$$

is homotopy commutative, there exists $f : S \to X'$ such that $f \circ q_2$ is homotopic to t and $f \circ q_1$ is homotopic to p_1'.

If Axiom 13 holds, then f admits a homotopy retraction r.

On the other hand, since $t \circ p_1 \simeq t \circ p_2 \simeq p_2' \circ \beta$, the square

$$\begin{array}{ccc} P & \xrightarrow{p_1} & X \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle t} \\ P' & \xrightarrow[p_2']{} & X' \end{array}$$

is homotopy commutative, so there exists $\Phi : S \to X'$, such that $\Phi \circ q_1$ is homotopic to p_2' and $\Phi \circ q_2$ is homotopic to t.

But $r \circ f \simeq id_S$, and so $\Phi \circ r \circ p_1' \simeq \Phi \circ r \circ f \circ q_1 \simeq \Phi \circ q_1 \simeq p_2'$.

Since the diagram

$$\begin{array}{ccc} P' & \xrightarrow{p_1'} & X' \\ {\scriptstyle p_2'}\downarrow & & \downarrow{\scriptstyle h'} \\ X' & \xrightarrow[h']{} & X' \end{array}$$

is a homotopy pull back, there exists $\psi : X' \to P'$ such that $p_1' \circ \psi \simeq id_{X'}$ and $p_2' \circ \psi \simeq id_{X'}$.

Therefore $\Phi \circ r \simeq id_{X'}$ and so p'_1 is homotopic to p'_2.

Now we show that h' is a homotopy monomorphism. Indeed, let $u, v : Z \to X'$ be such that $h' \circ u \simeq h' \circ v$. Since the diagram

$$\begin{array}{ccc} P' & \xrightarrow{p'_1} & X' \\ {\scriptstyle p'_2}\downarrow & & \downarrow{\scriptstyle h'} \\ X' & \xrightarrow[h']{} & X' \end{array}$$

is a homotopy pull back, there exists $\varphi : Z \to P'$ such that $p'_1 \circ \varphi \simeq u$ and $p'_2 \circ \varphi \simeq v$. But $p'_1 \simeq p'_2$, so u and v are homotopic and h' is a homotopy monomorphism. □

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