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The Ganea and Whitehead Variants of the Lusternik–Schnirelmann Category

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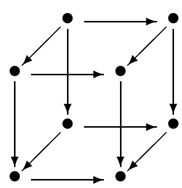
Abstract. The Lusternik–Schnirelmann category has been described in different ways. Two major ones, the first by Ganea, the second by Whitehead, are presented here with a number of variants. The equivalence of these variants relies on the axioms of Quillen's model category, but also sometimes on an additional axiom, the so-called "cube axiom".

The Lusternik–Schnirelmann category has been described in different ways. Two major ones by Ganea and by Whitehead, are presented here with a number of variants. The equivalence of these variants rely on the axioms of Quillen's model category, but also sometimes on the so-called cube axiom. The *cube axiom* is the following assertion: For any homotopy commutative diagram



if the bottom square is a homotopy push out and the four vertical squares are homotopy pull backs, then the top square is a homotopy push out. (See [7] for the original assertion.) This axiom, which is satisfied in the category of topological pointed spaces with the usual notion of homotopy, is also meaningful in Quillen's model categories (see [3] for more details), even if the constructions of homotopy pull backs and homotopy push outs must be done carefully via factorizations through fibrations and cofibrations (as this is done in [1, 2]).

In this paper, we work in the full subcategory $\mathbf{C}_{\mathrm{cf}}$ of cofibrant and fibrant objects of any pointed model category $\mathbf{C}$. The star $*$ will denote the initial and final object.

The basic example is the category $\mathbf{Top}^w$ of well-pointed topological spaces — a space X is *well pointed* if the map $* \to X$ is a closed cofibration. (See [8] for details about its model category structure.)

We will draw many *homotopy commutative diagrams*; such diagrams not only have objects and maps but also homotopies between every pair of composites of maps in the diagram with same source and target. We make this more precise with the following definition (it is actually that of [7], transposed in the model category setting):

Let $+$ denote the "track addition" of homotopies, and let $\sim$ denote the equivalence of homotopies. A *homotopy commutative diagram* in $\mathbf{C}_{\mathrm{cf}}$ is defined to consist of

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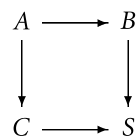
1. A set of objects and morphisms between them, together with the compositions of the morphisms.
2. For each pair $\alpha, \beta\colon A \to B$ in the diagram, a homotopy $H_{\alpha,\beta}\colon A \times I \to B$ from α to β such that:
   (a) $H_{\alpha,\alpha}$ is equivalent to the static homotopy (*i.e.,* the composite of $A \times I \to A$ with $\alpha\colon A \to B$);
   (b) if $\alpha, \beta, \gamma\colon A \to B$ then $H_{\alpha,\beta} + H_{\beta,\gamma} \sim H_{\alpha,\gamma}$;
   (c) if $\alpha\colon A \to B$, $\beta, \gamma\colon B \to C$ and $\epsilon\colon C \to D$, then

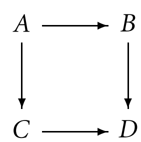
$$H_{\epsilon\circ\beta\circ\alpha,\epsilon\circ\gamma\circ\alpha} \sim \epsilon \circ H_{\beta,\gamma} \circ (\alpha \times I).$$

We now give the definitions of a *homotopy push out* and a *homotopy pull back* (also those of [7] transposed in the model category setting):

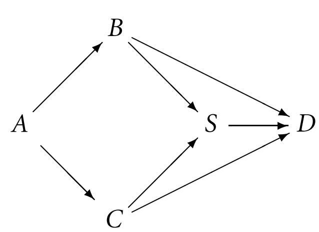
A homotopy commutative square



is a *homotopy push out* whenever for each other homotopy commutative square

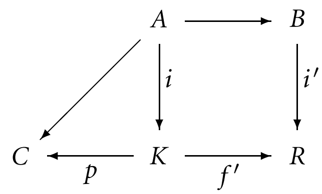


there is a map $S \to D$ (here called a *whisker map* as in [7]) and a homotopy commutative diagram (here called a *(homotopy) push out diagram*):

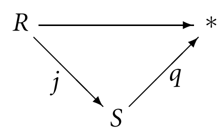


and the triplet made of the map $S \to D$ and the homotopies of the two triangles in the above diagram is unique up to homotopy and equivalences of homotopies. This notion dualizes to the one of *homotopy pull back*; here *dualize* means keeping the same diagrams but reversing all arrows.

Homotopy push outs and homotopy pull backs exist in $\mathbf{C}_{\mathrm{cf}}$. To build the homotopy push out of $f\colon A \to B$ and $g\colon A \to C$ in $\mathbf{C}_{\mathrm{cf}}$, choose any factorization $g = p \circ i$ where i is a cofibration and p is a fibration that is also a weak equivalence (p is a homotopy equivalence because both its source and target are in $\mathbf{C}_{\mathrm{cf}}$), then take the push out R (in $\mathbf{C}$) of f and i:



Finally (if R is not fibrant) choose any fibrant model S of R, *i.e.,* choose a factorization



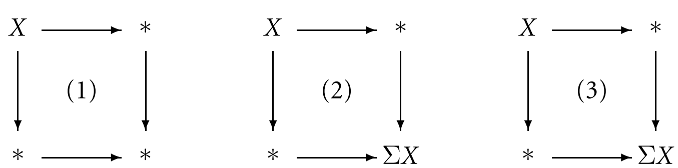
where j is a cofibration that is also a weak equivalence and q is a fibration, so S is in $\mathbf{C}_{\mathrm{cf}}$. The homotopy inverse of p composed with $j \circ f'$ gives us the map $C \to S$. The dual construction leads to the homotopy pull back. (See also [2] for other details.)

In $\mathbf{Top}^w$, the so-called "standard homotopy push out" $Z_{f,g}$ of [7] is a particular case of the above construction.

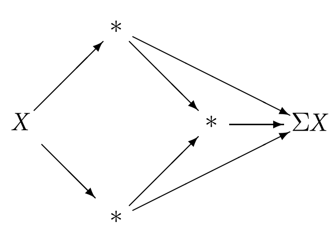
Warning: Because all diagrams come with homotopies, a homotopy push out is *not* a push out in the homotopy category Ho $\mathbf{C}$.

We will not write the homotopies explicitly in the sequel because in most cases, all we have to know is that they are there! However, it is important to keep in mind that all these homotopies are well defined (up to equivalences) and are *not* anything we can imagine. Here is an example.

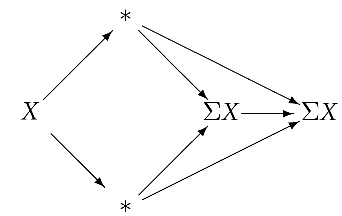
Let us consider the three homotopy commutative squares in $\mathbf{Top}^w$:



where (1) and (2) come with the static homotopy $H(x,t) = *$ and (3) comes with the homotopy $K(x,t) = [(x,t)]$. Only (3) is a homotopy push out. Indeed (1) is not a homotopy push out, because the diagram



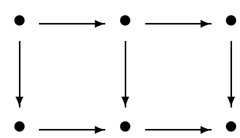
where the inside square is (1) and the outside square is (3) is not homotopy commutative — condition (b) is not satisfied when conditions (a) and (c) are. Nor is (2) a homotopy push out, because the diagram



where the inside square is (2) and the outside square is (3) cannot be homotopy commutative, whatever might be the map $\Sigma X \to \Sigma X$ (identity or the null map for instance).

The following property (here called the *prism lemma* as in [3]) is often used:

Lemma 1 *Assume the following diagram is homotopy commutative.*

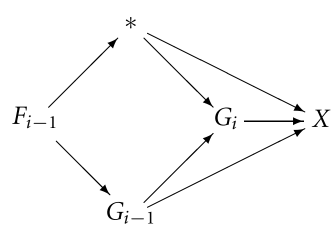


(i) *If the left square is a homotopy push out, then the outside rectangle is a homotopy push out if and only if the right square is a homotopy push out.*

(ii) *If the right square is a homotopy pull back, then the outside rectangle is a homotopy pull back if and only if the left square is a homotopy pull back.*

Warning: If the outside rectangle and the right square are homotopy push outs, the left one is *not* necessarily a homotopy push out. Dually, if the outside rectangle and the left square are homotopy pull backs, the right one is *not* necessarily a homotopy pull back.

For any X, the *Ganea construction* on X is the following sequence of homotopy push out diagrams ($i > 0$) starting with $G_0 \simeq *$:



where each map $F_{i-1} \to G_{i-1}$ is the homotopy fibre of $g_{i-1}\colon G_{i-1} \to X$ (which means that the outside square is a homotopy pull back).

Note: $G_1 \simeq \Sigma\Omega X$.

Let us define four versions of the Ganea category.

- We say that $\mathrm{G1cat}\, X \leq n$ if the following condition (G1) holds:

  (G1) The map $g_n\colon G_n \to X$ has a homotopy section.

G1cat X is the Ganea category defined in [4].

- We say that G2cat $X \leq n$ if the following condition (G2) holds:

  (G2) There exists a sequence of homotopy push outs

$$\begin{array}{ccc} Z_{i-1} & \longrightarrow & * \\ \downarrow & \text{h.p.o.} & \downarrow \\ Y_{i-1} & \longrightarrow & Y_i \end{array}$$

  $(0 < i \leq n)$ where $Y_0 \simeq *$, and X is a homotopy retract of Y_n.

Note that this condition implies that we have a map $y_n \colon Y_n \to X$, and by successive compositions with each $Y_{i-1} \to Y_i$, we have maps $y_{i-1} \colon Y_{i-1} \to X$. Each y_i is the whisker map induced by y_{i-1} and $* \to X$. Also note that, clearly, G2cat $G_n \leq n$.

- We say that G3cat $X \leq n$ if the following condition (G3) holds:

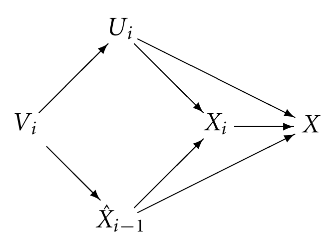
  (G3) Either $n = 0$ and $X \simeq *$, or $n > 0$ and there exists a homotopy push out

$$\begin{array}{ccc} M & \longrightarrow & * \\ \downarrow & \text{h.p.o.} & \downarrow \\ \hat{L} & \longrightarrow & L \end{array}$$

  where G3cat $\hat{L} \leq n - 1$ and X is a homotopy retract of L.

- We say that G4cat $X \leq n$ if the following condition (G4) holds:

  (G4) There exists a sequence of homotopy push out diagrams



  $(0 < i \leq n)$ where $X_0 \simeq *$, each map $\hat{x}_{i-1} \colon \hat{X}_{i-1} \to X$ factorizes through $x_{i-1} \colon X_{i-1} \to X$, each map $U_i \to X$ is null homotopic and $x_n \colon X_n \to X$ is the identity up to homotopy.

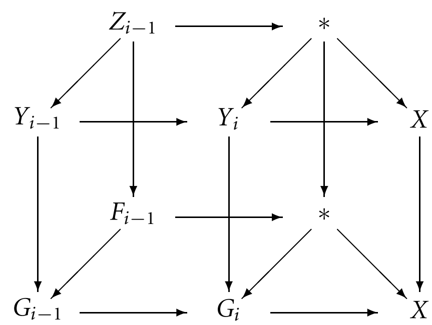
  This last definition appeared first in [5].

We now prove the equivalence of these conditions.

Proposition 2 *For any X we have* G1cat $X =$ G2cat $X =$ G3cat $X \leq$ G4cat X*, and if the cube axiom holds, also* G3cat $X =$ G4cat X.

Proof (G1) $\Rightarrow$ (G2). Obvious.

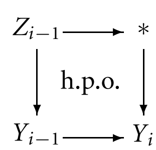
(G2) ⇒ (G1). We show inductively that $y_i\colon Y_i \to X$ factorizes through $g_i\colon G_i \to X$. Assume we have $\alpha_{i-1}\colon Y_{i-1} \to G_{i-1}$ with $g_{i-1}\alpha_{i-1}$ homotopic to y_{i-1}. Then we can construct a double push out diagram:



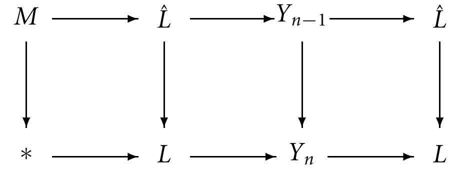
where $Z_{i-1} \to F_{i-1}$ is given as the whisker map and then $\alpha_i\colon Y_i \to G_i$ is given as the whisker map. The composite $g_i\alpha_i$ is homotopic to y_i by the universal property of the homotopy push out. So the inductive step is proven. At the end of the induction, we have $g_n\alpha_n$ homotopic to y_n, and as we have a homotopy section $\sigma_n\colon X \to Y_n$ of $y_n\colon Y_n \to X$, we get a homotopy section $\alpha_n\sigma_n$ for g_n.

(G2) ⇒ (G3). We prove this inductively on n. First note $\mathrm{G2cat}\, Y_i \le i$ for all i because each Y_i is a homotopy retract of itself. So assuming the step $n-1$ of the induction true, $\mathrm{G3cat}\, Y_{n-1} \le n-1$. To prove step n, we can choose M to be Z_{n-1} and $\hat{L}$ to be Y_{n-1}, so $L \simeq Y_n$.

(G3) ⇒ (G2). We prove this inductively on n. If $\mathrm{G3cat}\, X \le n$, we have $\mathrm{G3cat}\, \hat{L} \le n-1$, so by induction hypothesis $\mathrm{G2cat}\, \hat{L} \le n-1$, which means that we have a sequence of homotopy push outs

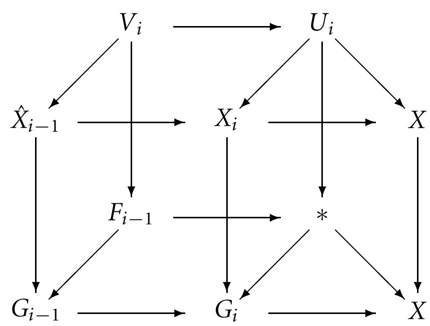


$(0 < i \le n-1)$ and $\hat{L}$ is a homotopy retract of Y_{n-1}. We can then construct Y_n as a homotopy push out in the following diagram where all squares are homotopy push outs:



So L appears to be a homotopy retract of Y_n and, as by hypothesis X is a homotopy rectract of L, we obtain that X is a homotopy retract of Y_n.

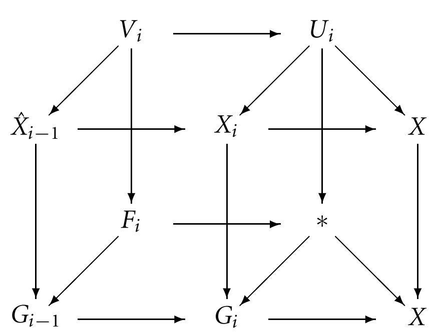
(G4) $\Rightarrow$ (G1). We prove that $x_i\colon X_i \to X$ factorizes through $g_i\colon G_i \to X$. This is true for $i = 0$ as $X_0 \simeq *$. Now, assuming that $x_{i-1}\colon X_{i-1} \to X$ factorizes through $g_{i-1}\colon G_{i-1} \to X$, then also $\hat{x}_{i-1}$ factorizes through g_{i-1} and we can construct the following double push out diagram:



where the map $V_i \to F_{i-1}$ is the whisker map to the homotopy pull back, and then the map $\beta_i\colon X_i \to G_i$ is the whisker map from the homotopy push out. The composite $g_i\beta_i$, which is a whisker map of the homotopy push out, is homotopic to x_i by the universal property of the homotopy push out. So the inductive step is proven. And at the end of the induction we get $g_n\beta_n$ homotopic to x_n which is the identity up to homotopy, so β_n is a homotopy section for g_n.

Finally, if the cube axiom is satisfied, we prove:

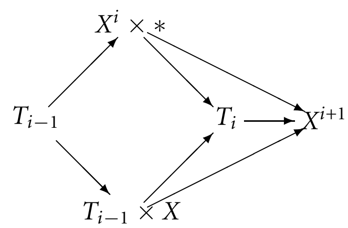
(G1) $\Rightarrow$ (G4). We can construct the following double push out diagram, with a descending induction on i:



For $i = n$, we set $X_n = X$, $X_n \to X$ is the identity and $X_n \to G_n$ is the homotopy section of $g_n\colon G_n \to X$. We take homotopy pull backs to construct the four vertical squares of the cube. The upper square of the cube is a homotopy push out by the cube axiom. We can choose $x_{i-1}\colon X_{i-1} \to X$ to be either $\hat{X}_{i-1} \to X$ or $g_{i-1}\colon G_{i-1} \to X$ to proceed to the next step of the induction. Actually all following steps will be trivial if

x_{i-1} is choosen to be g_{i-1}, because then the map $X_j \to G_j$ will be the identity for all $j < i$. ∎

For any X, the *Whitehead construction* on X is the following sequence of homotopy push out diagrams ($i > 0$) starting with $T_0 \simeq *$:



Note that the outside square is a homotopy pull back. Use the prism lemma in the following diagram:

$$\begin{array}{ccccc} T_{i-1} & \longrightarrow & X^i & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ T_{i-1} \times X & \longrightarrow & X^i \times X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ T_{i-1} & \longrightarrow & X^i & \longrightarrow & * \end{array}$$

Note: $T_1 \simeq X \vee X$.

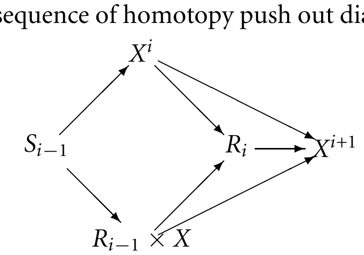
Let us now give three versions of the Whitehead category.

- We say that W1cat $X \leq n$ if the following condition (W1) holds:

  (W1) The diagonal $\Delta\colon X \to X^{n+1}$ factorizes through $t_n\colon T_n \to X^{n+1}$ up to homotopy.

  W1cat X is the Whitehead category defined in [10].
- We say that W2cat $X \leq n$ if the following condition (W2) holds:

  (W2) There exists a sequence of homotopy push out diagrams:

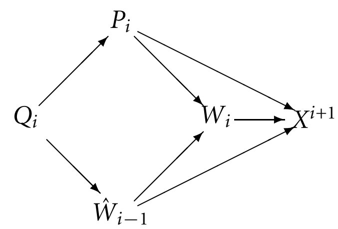


$(0 < i \leq n)$ where $R_0 \simeq *$ and the diagonal $\Delta\colon X \to X^{n+1}$ factorizes through $r_n\colon R_n \to X^{n+1}$ up to homotopy. (Note that S_{i-1} must not be R_{i-1}, so R_i is not T_i.)

We do not know (and therefore ask) if there exists some condition (W3) corresponding to (G3).

- We say that $\mathrm{W4cat}\, X \leq n$ if the following condition (W4) holds:

(W4) There exists a sequence of homotopy push out diagrams:



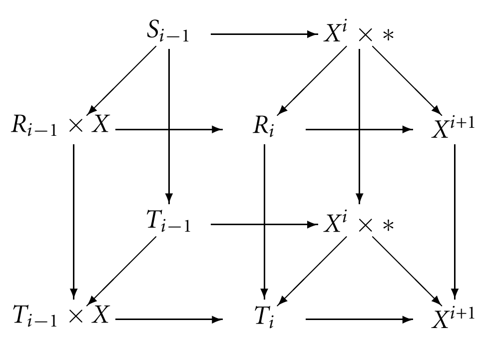
$(0 < i \leq n)$ where $W_0 \simeq *$, the map $w_i\colon W_i \to X^{i+1}$ is induced by $\hat{W}_{i-1} \to X^{i+1}$, which factorizes through $w_{i-1} \times \mathrm{id}_X\colon W_{i-1} \times X \to X^{i+1}$, and by $P_i \to X^{i+1}$, which factorizes through $X^i \times * \to X^{i+1}$. The map $w_n\colon W_n \to X^{n+1}$ is the diagonal $X \to X^{n+1}$ up to homotopy.

We now prove the equivalence of these conditions.

Proposition 3 *For any X we have* $\mathrm{W1cat}\, X = \mathrm{W2cat}\, X \leq \mathrm{W4cat}\, X$*, and if the cube axiom holds, also* $\mathrm{W2cat}\, X = \mathrm{W4cat}\, X$.

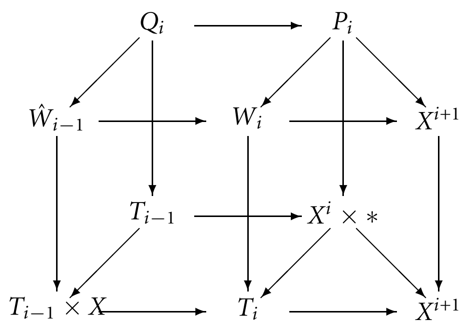
Proof (W1) $\Rightarrow$ (W2). Obvious.

(W2) $\Rightarrow$ (W1). We show inductively that $r_i\colon R_i \to X^{i+1}$ factorizes through $t_i\colon T_i \to X^{i+1}$. Assume we have $\gamma_{i-1}\colon R_{i-1} \to T_{i-1}$ with $t_{i-1}\gamma_{i-1}$ homotopic to r_{i-1}. Then we can construct a homotopy commutative diagram:



where $S_{i-1} \to T_{i-1}$ is given as the whisker map and then $\gamma_i\colon R_i \to T_i$ is given as the whisker map. The composite $t_i\gamma_i$ is homotopic to r_i by the universal property of the homotopy push out. So the inductive step is proven. At the end of the induction, we have $t_n\gamma_n$ homotopic to r_n, and as the diagonal $\Delta\colon X \to X^{n+1}$ factorizes through r_n, it also factorizes through t_n.

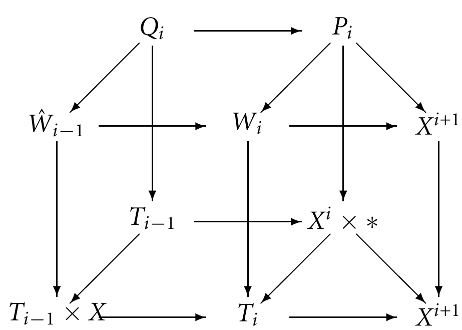
(W4) $\Rightarrow$ (W1). We prove that $w_i\colon W_i \to X^{i+1}$ factorizes through $t_i\colon T_i \to X^{i+1}$. This is true for $i = 0$ as $W_0 \simeq *$. Now assuming that $w_{i-1}\colon W_{i-1} \to X^i$ factorizes through $t_{i-1}\colon T_{i-1} \to X^i$, then $w_{i-1} \times \mathrm{id}_X\colon W_{i-1} \times X \to X^{i+1}$ factorizes through $t_{i-1} \times \mathrm{id}_X\colon T_{i-1} \times X \to X^{i+1}$ and we can construct the following double push out diagram:



where the map $Q_i \to T_{i-1}$ is the whisker map to the homotopy pull back, and then the map $\delta_i\colon W_i \to T_i$ is the whisker map from the homotopy push out. The composite $t_i\delta_i$, which is a whisker map of the homotopy push out, is homotopic to w_i by the universal property of the homotopy push out. So the inductive step is proven. And at the end of the induction we get $t_n\delta_n$ homotopic to w_n, and since the diagonal factorizes through w_n by hypothesis, it factorizes also through t_n.

Finally, if the cube axiom is satisfied, we prove:

(W1) $\Rightarrow$ (W4). We can construct the following double push out diagram, with a descending induction on i:



For $i = n$, $w_n\colon W_n \to X^{n+1}$ is the diagonal and $W_n \to T_n$ is the homotopy lifting of the diagonal through $t_n\colon T_n \to X^{n+1}$ which exists by hypothesis. We take homotopy pull backs to construct the four vertical squares of the cube. The upper square of the cube is a homotopy push out by the cube axiom. We can choose $w_{i-1}\colon W_{i-1} \to X^i$ to be $t_{i-1}\colon T_{i-1} \to X^i$. So the map $\hat{W}_{i-1} \to X^{i+1}$ factorizes through $w_{i-1} \times id_X =$

$t_{i-1} \times id_X$. We can then proceed to the next step of the induction. Actually, all steps except the first one are trivial, because for all $i < n$, the map $W_i \to T_i$ will be the identity. ■

To finish we prove the equivalence of the Ganea and Whitehead categories when the cube axiom holds.

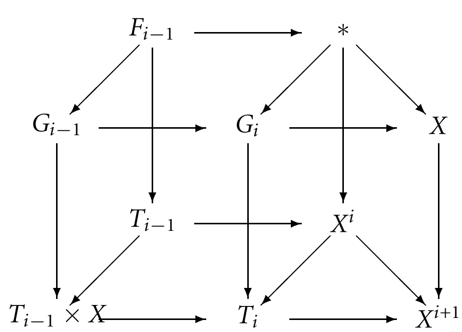
Theorem 4 *For any* X*, we have* $\mathrm{W1cat}\,X \leq \mathrm{G1cat}\,X$ *and if the cube axiom holds,* $\mathrm{W1cat}\,X = \mathrm{G1cat}\,X$.

Proof We prove inductively that we have a homotopy commutative diagram:

$$\begin{array}{ccc} G_i & \xrightarrow{g_i} & X \\ {\scriptstyle \epsilon_i}\downarrow & & \downarrow{\scriptstyle \Delta} \\ T_i & \xrightarrow[t_i]{} & X^{i+1} \end{array}$$

which, moreover, is a homotopy pull back when the cube axiom holds.

For $i = 0$, the above square exists. Assume it exists at the step $i - 1$. We can construct the following double push out diagram:



The homotopy commutative front rectangle

$$\begin{array}{ccc} G_{i-1} & \xrightarrow{g_{i-1}} & X \\ {\scriptstyle (\epsilon_{i-1}, g_{i-1})}\downarrow & & \downarrow{\scriptstyle \Delta} \\ T_{i-1} \times X & \xrightarrow[t_{i-1} \times \mathrm{id}]{} & X^i \times X \end{array}$$

comes from the induction hypothesis. The map $F_{i-1} \to T_{i-1}$ is the whisker map to the homotopy pull back and the map $\epsilon_i \colon G_i \to T_i$ is the whisker map from the homotopy push out. We get the right vertical square which is homotopy commutative, and so the induction step is done.

At the end of the induction we obtain the square

$$\begin{array}{ccc} G_n & \xrightarrow{g_n} & X \\ {\scriptstyle \epsilon_n}\downarrow & & \downarrow{\scriptstyle \Delta} \\ T_n & \xrightarrow[t_n]{} & X^{n+1} \end{array}$$

So if g_n has a homotopy section, Δ factorizes through t_n.

Now, if the cube axiom holds, we prove inductively that each square

$$\begin{array}{ccc} G_i & \longrightarrow & X \\ \downarrow & & \downarrow \\ T_i & \longrightarrow & X^{i+1} \end{array}$$

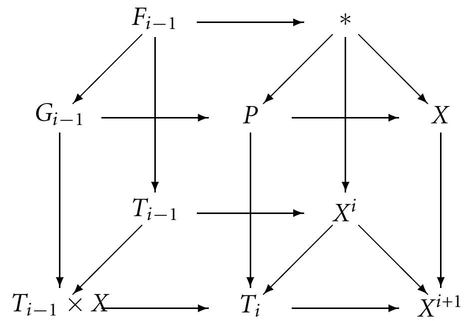
is a homotopy pull back. Indeed it is true at step 0; assume it is true at step $i-1$. Using this hypothesis and the prism lemma in the following diagram

$$\begin{array}{ccccc} G_{i-1} & \longrightarrow & X & & \\ \downarrow & & \downarrow{\scriptstyle \Delta} & & \\ T_{i-1} \times X & \longrightarrow & X^i \times X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ T_{i-1} & \longrightarrow & X^i & \longrightarrow & * \end{array}$$

we get a homotopy pull back

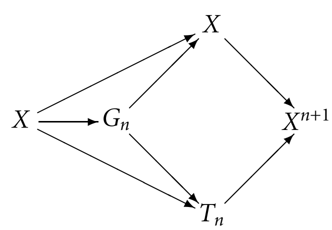
$$\begin{array}{ccc} G_{i-1} & \longrightarrow & X \\ \downarrow & \text{h.p.b.} & \downarrow \\ T_{i-1} \times X & \longrightarrow & X^i \times X \end{array}$$

Let P be the homotopy pull back of $t_i \colon T_i \to X^{i+1}$ and $\Delta \colon X \to X^{i+1}$. We can construct the following homotopy commutative diagram:



where the map $G_{i-1} \to P$ is the whisker map. As the front rectangle and the right square of the diagram are homotopy pull backs, so are the front square and the right square of the inside cube by the prism lemma. Moreover, as the top and bottom lozenges are homotopy pull backs too, so are the rear and left squares of the diagram by the prism lemma again. Thus, all the vertical faces of the inside cube are homotopy pull backs, and as the bottom face is a homotopy push out, so is the top face of the cube; this means that $G_i \simeq P$ and the inductive step is proven.

Finally, if Δ factorizes through t_n, then g_n has a section which is the whisker map induced by the identity on X and the lifting map $X \to T_n$:



■

Now what if the cube axiom does *not* hold? Let us look to the opposite category of topological spaces — we thank the referee for the suggestion — where G1cat is what is usually called the *cocategory*. In this category, $\mathrm{W1cat}\, X \le 1$ means that X is an H-space. It was proved by James (see [6]) that if X is an H-space, then there is a homotopy retraction $r\colon \Omega\Sigma X \to X$ for the natural map $\rho\colon X \to \Omega\Sigma X$ (see [6]); so $\mathrm{G1cat}\, X \le 1$. So when X is an H-space, $\mathrm{W1cat}\, X = \mathrm{G1cat}\, X$, despite the fact that the cube axiom does not hold in the opposite category of topological spaces. Moreover, if X is an H-space, we have also a homotopy fibration $X \to X \bowtie X \to \Sigma X$ where $j\colon X \to X \bowtie X$ is the inclusion of X into the join of X with itself (this is a null homotopic map) and $q\colon X \bowtie X \to \Sigma X$ is the Hopf construction (see [9]); so $\mathrm{G4cat}\, X \le 1$, too. So when X is an H-space, $\mathrm{G1cat}\, X = \mathrm{G4cat}\, X$, despite the fact that the cube axiom does not hold in the opposite category of topological spaces. We do not know, and ask, if these equalities also hold for $\mathrm{W1cat}\, X$ greater than 1, even when the cube axiom does not hold!

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