Bull. of the Belgian Math. Soc., volume 20, issue 4 (2013).

WHEN DOES SECAT EQUAL RELCAT ?

JEAN-PAUL DOERAENE AND MOHAMMED EL HAOUARI

Abstract. In [3] the authors introduced a *relative category* for a map that differs from the *sectional category* by just one. The relative category has specific properties (for instance a homotopy pushout does not increase it) which make it a convenient tool to study the sectional category. The question to know when secat equals relcat arises. We give here some sufficient conditions. Applications are given to the *topological complexity*, which is nothing but the sectional category of the diagonal.

In [3], we have introduced an approximation of James' *sectional category* of a map that we called *relative category*. For any continuous map $\iota : A \to X$, we have $\mathrm{secat}\,(\iota) \leqslant \mathrm{relcat}\,(\iota) \leqslant \mathrm{secat}\,(\iota) + 1$. It is an important information to know whether $\mathrm{secat}\,(\iota) = \mathrm{relcat}\,(\iota)$. For instance, when the equality holds, if C is the homotopy cofibre of ι, we have $\mathrm{cat}\,(C) \leqslant \mathrm{secat}\,(\iota) \leqslant \mathrm{cat}\,(X)$, see Corollary 5. For the null map $0_X : * \to X$, the equality is trivial: $\mathrm{secat}\,(0_X) = \mathrm{relcat}\,(0_X) = \mathrm{cat}\,(X)$. Here we establish the equality in three cases: the homotopy fibre of a map that has a homotopy section, see Proposition 8; the diagonal map of a connected CW H-space, see Theorem 11; and a $(q-1)$-connected map $\iota : A \to X$ where A is CW with $\dim A < (\mathrm{secat}\,(\iota) + 1)q - 1$, see Theorem 14.

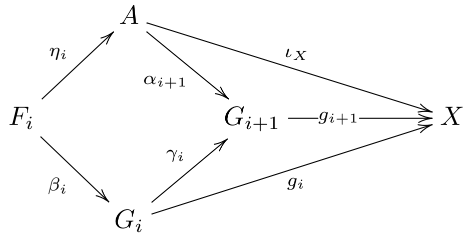
We work indifferently in the category of topological spaces **Top** or in the category of well-pointed topological spaces $\mathbf{Top^w}$ (*well-pointed* means that the inclusion of the base point is a closed cofibration) [9]. We will denote these categories ambigously by $\mathcal{T}$. However for most applications (for instance when we speak of homotopy fibre or cofibre) we need the category to be pointed (the zero object will be denoted by $*$). All constructions are made 'up to homotopy'.

We use the same notations as in [3]. The homotopy pullback of maps $f\colon A \to B$ and $g\colon C \to B$ is denoted by $A \times_B C$. If there are maps $p\colon D \to A$ and $q\colon D \to C$ such that $f \circ p \simeq g \circ q$, the 'whisker' map $D \to A \times_B C$ induced by the homotopy pullback is denoted by (p, q). The homotopy pushout of maps $v\colon U \to V$ and $w\colon U \to W$ is denoted by $V \vee_U W$. If there are maps $y\colon V \to X$ and $z\colon W \to X$ such that $y \circ v \simeq z \circ w$, the 'whisker' map $V \vee_U W \to X$ induced by the homotopy pushout is denoted by (y, z). If $W \simeq *$, then $V \vee_U *$ is the homotopy cofibre of v and is denoted by V/U. Finally the join of f and g is the whisker map $(f, g)\colon A \vee_P C \to B$ where $P \simeq A \times_B C$; $A \vee_P C$ is denoted by $A \bowtie_B C$. For basic definitions and properties about homotopy pullbacks and pushouts, we refer to [7] or [2].

2010 *Mathematics Subject Classification.* 55M30.
Key words and phrases. Ganea fibration, sectional category, topological complexity.

1. Sectional and relative categories

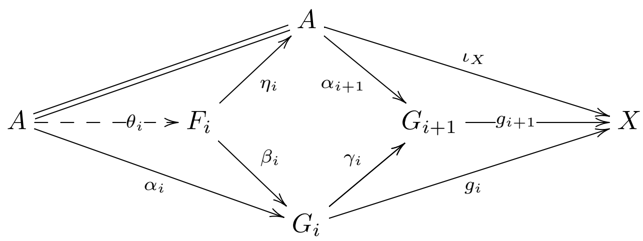
Definition 1. For any map $\iota_X\colon A \to X$ of $\mathcal{T}$, the *Ganea construction* of ι_X is the following sequence of homotopy commutative diagrams ($i \geqslant 0$):



where the outside square is a homotopy pullback, the inside square is a homotopy pushout and the map $g_{i+1} = (g_i, \iota_X)\colon G_{i+1} \to X$ is the whisker map induced by this homotopy pushout. The iteration starts with $g_0 = \iota_X\colon A \to X$.

We denote G_i by $G_i(\iota_X)$, or by $G_i(X, A)$. If $\mathcal{T}$ is pointed, we write $G_i(X) = G_i(X, *)$.

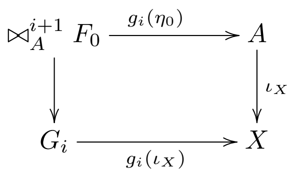
The sequence of homotopy commutative diagrams above extends to:



where $\alpha_0 = \mathrm{id}_A$. Since $g_i \circ \alpha_i \simeq \iota_X$, the outside square commutes up to homotopy and the homotopy pullback F_i induces the whisker map $\theta_i = (\alpha_i, \mathrm{id}_A)\colon A \to F_i$. Notice also that $\gamma_i \circ \alpha_i \simeq \alpha_{i+1}$.

Proposition 2. *For any map $\iota_X\colon A \to X$ in $\mathcal{T}$, we have $G_i(\iota_X) \simeq \bowtie_X^{i+1} A$, i.e. the $(i+1)$-fold join of A over X, and $F_i(\iota_X) \simeq \bowtie_A^{i+1} F_0(\iota_X)$.*

Proof. By definition, $G_i \simeq \bowtie_X^{i+1} A$. From the Join theorem, see [1], which asserts that, roughly speaking, the join of homotopy pullbacks is a homotopy pullback, we deduce that the following square is a homotopy pullback:



This means that $F_i \simeq \bowtie_A^{i+1} F_0$. □

Definition 3. Let $\iota_X\colon A \to X$ be a map of $\mathcal{T}$.

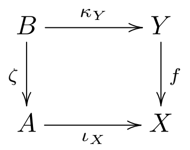
1) The *sectional category* of ι_X is the least integer n such that the map $g_n\colon G_n(\iota_X) \to X$ has a homotopy section, i.e. there exists a map $\sigma\colon X \to G_n(\iota_X)$ such that $g_n \circ \sigma \simeq \mathrm{id}_X$.

2) The *relative category* of ι_X is the least integer n such that the map $g_n\colon G_n(\iota_X) \to X$ has a homotopy section σ and $\sigma \circ \iota_X \simeq \alpha_n$.

We denote the sectional category by $\mathrm{secat}\,(\iota_X)$ or $\mathrm{secat}\,(X, A)$, and the relative category by $\mathrm{relcat}\,(\iota_X)$ or $\mathrm{relcat}\,(X, A)$. If $\mathcal{T}$ is pointed with $*$ as zero object, we write $\mathrm{cat}\,(X) = \mathrm{secat}\,(X, *) = \mathrm{relcat}\,(X, *)$. The integer $\mathrm{cat}\,(X)$ is the 'normalized' version of the Lusternik-Schnirelmann category.

The following basic facts about secat and relcat are proved in [3]:

Proposition 4. *Suppose we are given any homotopy commutative diagram in* $\mathcal{T}$*:*



1) If f has a homotopy section, then $\mathrm{secat}\,(\iota_X) \leqslant \mathrm{secat}\,(\kappa_Y)$.

2) If f has a homotopy section s, ζ has a homotopy section t, and $s\circ\iota_X \simeq \kappa_Y\circ t$, then $\mathrm{relcat}\,(\iota_X) \leqslant \mathrm{relcat}\,(\kappa_Y)$.

3) If the square is a homotopy pullback, then

$$\mathrm{secat}\,(\kappa_Y) \leqslant \mathrm{secat}\,(\iota_X) \text{ and } \mathrm{relcat}\,(\kappa_Y) \leqslant \mathrm{relcat}\,(\iota_X).$$

4) If the square is a homotopy pushout, then $\mathrm{relcat}\,(\iota_X) \leqslant \mathrm{relcat}\,(\kappa_Y)$.

5) If f and ζ have homotopy inverses, then

$$\mathrm{secat}\,(\iota_X) = \mathrm{secat}\,(\kappa_Y) \text{ and } \mathrm{relcat}\,(\iota_X) = \mathrm{relcat}\,(\kappa_Y).$$

Two particular cases (of statements 1 and 4) are worth to be remarked: For any map $\iota_X\colon A \to X$, we have $\mathrm{secat}\,(\iota_X) \leqslant \mathrm{cat}\,(X)$ and $\mathrm{cat}\,(X/A) \leqslant \mathrm{relcat}\,(\iota_X)$.

The following immediate consequence inlights the importance of knowing when sectional and relative categories coincide:

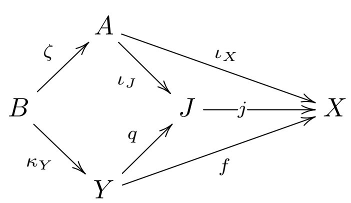
Corollary 5. *For any map $\iota_X\colon A \to X$ with homotopy cofibre X/A, if* $\mathrm{secat}\,\iota_X = \mathrm{relcat}\,\iota_X$*, then*

$$\mathrm{cat}\,(X/A) \leqslant \mathrm{secat}\,(\iota_X) \leqslant \mathrm{cat}\,(X).$$

Recall that in general $\mathrm{cat}\,(X/A) \leqslant \mathrm{cat}\,(X)+1$. It is important to note that if the sectional and relative categories of a map are equal, the category of its homotopy cofibre cannot be greater than the category of its target.

The following other consequence of Proposition 4 will be useful:

Proposition 6. *If $\iota_X\colon A \to X$ and $f\colon Y \to X$ are maps of $\mathcal{T}$, consider the following join construction:*



where the outside square is a homotopy pullback, the inside square is a homotopy pushout, and the map $j = (f, \iota_X)\colon J \to X$ is the whisker map induced by the homotopy pushout. We have

$$\mathrm{relcat}\,(\iota_J) \leqslant \mathrm{relcat}\,(\kappa_Y) \leqslant \mathrm{relcat}\,(\iota_X).$$

Moreover, if f has a homotopy section, then

$$\mathrm{relcat}\,(\iota_J) = \mathrm{relcat}\,(\kappa_Y) = \mathrm{relcat}\,(\iota_X).$$

Proof. The inequalities are direct applications of Proposition 4, statements 3 and 4.

If s is a homotopy section of f, the Prism lemma (see [2] for instance) gives the two homotopy pullbacks:

$$\begin{array}{ccccc} A & \xrightarrow{t} & B & \xrightarrow{\zeta} & A \\ {\scriptstyle \iota_X}\downarrow & & {\scriptstyle \kappa_Y}\downarrow & & \downarrow{\scriptstyle \iota_X} \\ X & \xrightarrow[s]{} & Y & \xrightarrow[f]{} & X \end{array}$$

and $\zeta \circ t \simeq \mathrm{id}_A$. We have $j \circ q \circ s \simeq f \circ s \simeq \mathrm{id}_X$, so $q \circ s$ is a homotopy section of j. Also we have $q \circ s \circ \iota_X \simeq q \circ \kappa_Y \circ t \simeq \iota_J \circ \zeta \circ t \simeq \iota_J$, and we obtain $\mathrm{relcat}\,(\iota_X) \leqslant \mathrm{relcat}\,(\iota_J)$ by Proposition 4, statement 2. □

An interesting particular case of Proposition 6 is this one:

Corollary 7. *Let $i\colon F \to E$ be the homotopy fibre of $f\colon E \to B$ and E/F be the homotopy cofibre of i. Then:*

$$\mathrm{cat}\,(E/F) \leqslant \mathrm{relcat}\,(i) \leqslant \mathrm{cat}\,(B).$$

2. Comparing sectional and relative categories

We obtain a first sufficient condition for the equality of sectional and relative categories of a map:

Proposition 8. *Let $i\colon F \to E$ be the homotopy fibre of $f\colon E \to B$. If f has a homotopy section then* $\mathrm{cat}\,(E/F) = \mathrm{relcat}\,(i) = \mathrm{cat}\,(B) = \mathrm{secat}\,(i)$.

Proof. The first two equalities are direct applications of Proposition 6. Proposition 4, statements 1 and 3, imply the third equality. □

Example 9. The map $\mathrm{in}_1 = (\mathrm{id}_A, 0)\colon A \to A \times B$ is the (homotopy) fibre of $\mathrm{pr}_2\colon A \times B \to B$, thus $\mathrm{cat}\,((A \times B)/A) = \mathrm{secat}\,(\mathrm{in}_1) = \mathrm{relcat}\,(\mathrm{in}_1) = \mathrm{cat}\,(B)$.

For any X in $\mathcal{T}$, and $m \geqslant 2$, recall from [8], that the *higher topological complexity* $\mathrm{TC}_m(X)$ is defined as $\mathrm{TC}_m(X) = \mathrm{secat}\,(\Delta_m)$, i.e. it is the sectional category of the diagonal $\Delta_m\colon X \to X^m$. Farber's topological complexity $\mathrm{TC}\,(X) = \mathrm{TC}_2(X)$. (Originally, there was a shift by one; we use here the 'normalized' definition.)

Proposition 10. *For any X in $\mathcal{T}$, and $m \geqslant 2$, we have*

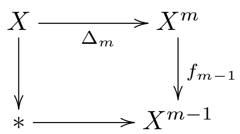
$$\mathrm{cat}\,(X^{m-1}) \leqslant \mathrm{TC}_m(X) \leqslant \mathrm{cat}\,(X^m).$$

Proof. Follows from Proposition 4, see [3]. □

Theorem 11. *Let X be a connected, CW H-space. For any $m \geqslant 2$, we have*

$$\mathrm{cat}\,(X^m/X) = \mathrm{TC}_m(X) = \mathrm{secat}\,(\Delta_m) = \mathrm{relcat}\,(\Delta_m) = \mathrm{cat}\,(X^{m-1}).$$

Proof. It is shown in [6] that for a connected CW H-space X, there is a homotopy pullback:



and f_{m-1} has an obvious homotopy section. Thus we obtain the desired equalities by Proposition 8. □

Our own contribution here is the equality $\mathrm{secat}\,(\Delta_m) = \mathrm{relcat}\,(\Delta_m)$. The equality $\mathrm{secat}\,(\Delta_m) = \mathrm{cat}\,(X^{m-1})$ is shown in [6] and the equality $\mathrm{cat}\,(X^m/X) = \mathrm{secat}\,(\Delta_m)$ is shown in [4].

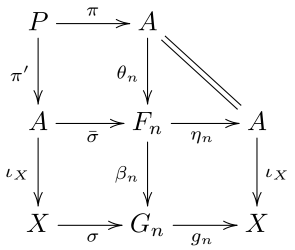
We proved the next result indirectly in [3]. We give here a direct proof for convenience.

Proposition 12. *For any map $\iota_X\colon A \to X$ of $\mathcal{T}$, we have:*

$$\mathrm{secat}\,(\iota_X) \leqslant \mathrm{relcat}\,(\iota_X) \leqslant \mathrm{secat}\,(\iota_X) + 1.$$

Proof. Let $\mathrm{secat}\,(\iota_X) \leqslant n$. Consider any homotopy section $\sigma\colon X \to G_n$ of $g_n\colon G_n \to X$ and let $\sigma^+ = \gamma_n \circ \sigma$. Following the proof of Proposition 6, we have that σ^+ is a homotopy section of g_{n+1} and $\sigma^+ \circ \iota_X \simeq \alpha_{n+1}$. We have obtained that $\mathrm{relcat}\,(\iota_X) \leqslant n + 1$. □

Let be given any map $\iota_X\colon A \to X$ with $\mathrm{secat}\,(\iota_X) \leqslant n$ and any homotopy section $\sigma\colon X \to G_n$ of $g_n\colon G_n \to X$. Consider the following homotopy pullbacks:



By the Prism lemma, we know that the homotopy pullback of σ and β_n is indeed A, and that $\eta_n \circ \bar{\sigma} \simeq \mathrm{id}_A$. Also notice that $\pi \simeq \pi'$ since $\pi \simeq \eta_n \circ \theta_n \circ \pi \simeq \eta_n \circ \bar{\sigma} \circ \pi' \simeq \pi'$.

Proposition 13. *Let be given any map $\iota_X\colon A \to X$ with* $\mathrm{secat}\,(\iota_X) \leqslant n$ *and any homotopy section $\sigma\colon X \to G_n(\iota_X)$ of $g_n\colon G_n(\iota_X) \to X$. With the same definitions and notations as above, the following conditions are equivalent:*

(i) $\sigma \circ \iota_X \simeq \alpha_n$.
(ii) π *has a homotopy section.*
(iii) π *is a homotopy epimorphism.*
(iv) $\theta_n \simeq \bar{\sigma}$.

Proof. We have the following sequence of implications:

(i) $\Longrightarrow$ (ii): Since $\sigma \circ \iota_X \simeq \alpha_n \simeq \beta_n \circ \theta_n \circ \mathrm{id}_A$, we have a whisker map $(\iota_X, \mathrm{id}_A)\colon A \to P$ induced by the homotopy pullback P which is a homotopy section of π.

(ii) $\Longrightarrow$ (iii): Obvious.

(iii) $\implies$ (iv): We have $\theta_n \circ \pi \simeq \bar{\sigma} \circ \pi$ since $\pi \simeq \pi'$. Thus $\theta_n \simeq \bar{\sigma}$ since π is a homotopy epimorphism.

(iv) $\implies$ (i): We have $\sigma \circ \iota_X \simeq \beta_n \circ \bar{\sigma} \simeq \beta_n \circ \theta_n \simeq \alpha_n$. □

Theorem 14. *Let be given a CW-complex A and a $(q-1)$-connected map $\iota_X\colon A \to X$. If $\dim A < (\mathrm{secat}\, \iota_X + 1)q - 1$ then $\mathrm{secat}\, \iota_X = \mathrm{relcat}\, \iota_X$.*

Proof. Recall that g_i is the $(i+1)$-fold join of ι_X. Thus by [7], Theorem 47, we obtain that, for each $i \geqslant 0$, $g_i : G_i \to X$ is $(i+1)q-1$-connected. As g_i and η_i have the same homotopy fibre, the Five lemma implies that $\eta_i\colon F_i \to A$ is $(i+1)q-1$-connected, too. By [10], Theorem IV.7.16, this means that for every CW-complex K with $\dim K < (i+1)q-1$, η_i induces a one-to-one correspondence $[K, F_i] \to [K, A]$. Since θ_n and $\bar{\sigma}$ are both homotopy sections of η_n, we obtain $\theta_n \simeq \bar{\sigma}$, and Proposition 13 implies the desired result. □

Example 15. Let $\iota\colon S^r \to S^m$ with $r \geqslant m$. If $r < 2m-1$, then $\mathrm{relcat}\,(\iota) = \mathrm{secat}\,(\iota)$; this is 1 except for the identity for which it is 0. In particular this means that $\alpha_1\colon S^r \to S^r \bowtie_{S^m} S^r$ factorizes through ι up to homotopy.

Example 16. Let h be any of the Hopf maps $S^3 \to S^2$, $S^7 \to S^4$ and $S^{15} \to S^8$. Since they have a target of category 1 and a homotopy cofibre of category 2, we have $\mathrm{secat}\, h = 1$ while $\mathrm{relcat}\, h = 2$. This is a conterexample wich illustrates that the inequality in the hypothesis of Theorem 14 is sharp, because in the three cases we have exactly $\dim A = (\mathrm{secat}\, h + 1)q - 1$.

In [3], we have introduced the *complexity of a map* $\iota_X\colon A \to X$; we write $\mathrm{TC}\,(\iota) = \mathrm{secat}\,(\mathrm{id}_A, \iota_X)$ where $(\mathrm{id}_A, \iota_X)\colon A \to A \times X$ is the whisker map induced by the homotopy pullback. In particular the complexity of the null map $* \to X$ is $\mathrm{cat}\,(X)$ (see Example 9) and the complexity of id_X is $\mathrm{secat}\,(\Delta) = \mathrm{TC}\,(X)$. We will also write $\mathrm{relTC}\,(\iota_X) = \mathrm{relcat}\,(\mathrm{id}_A, \iota_X)$.

Proposition 17. *For any map $\iota_X\colon A \to X$ in $\mathcal{T}$, we have:*

$$\mathrm{cat}\,(X) \leqslant \mathrm{TC}\,(\iota_X) \leqslant \mathrm{TC}\,(X) \leqslant \mathrm{cat}\,(X \times X).$$

Proof. Follows from Proposition 4, see [3]. □

Applying Theorem 14 to topological complexity, we obtain:

Corollary 18. *Let be given any map $\iota_X\colon A \to X$ between CW-complexes, A connected and X $(q-1)$-connected. If $\dim A < (\mathrm{TC}\,(\iota_X) + 1)q - 1$, then*

$$\mathrm{cat}\,((A \times X)/A) \leqslant \mathrm{relTC}\,(\iota_X) = \mathrm{TC}\,(\iota_X) \leqslant \mathrm{cat}\,(A \times X)$$

where $(A \times X)/A$ is the homotopy cofibre of (id_A, ι_X).

Proof. With the hypothesis, (id_A, ι_X) is $(q-1)$-connected, and we may apply Theorem 14 to obtain the equality. This implies the inequalities by Corollary 5. □

The first inequality is proved in [4] for the particular case $\iota_X = \mathrm{id}_X$.

Example 19. Consider the Hopf fibration $S^7 \to S^4$ and factor by the action of S^1 on S^7 to get $\iota\colon \mathbb{C}P^3 \to S^4$. We have shown in [3] that $\mathrm{TC}\,(\iota) = 2$. We have $\dim \mathbb{C}P^3 = 6 < 3.4 - 1 = (\mathrm{TC}\,(\iota) + 1).q - 1$, so $\mathrm{relTC}\,(\iota) = \mathrm{TC}\,(\iota) = 2$.

Example 20. More generally assume A is a connected CW-complex and consider any map $\iota\colon A \to S^m$. We have $\mathrm{TC}(\iota) \geqslant \mathrm{cat}(S^m) = 1$ and S^m is $(m-1)$-connected. Thus if $\dim A < 2m-1$, we have $\mathrm{relTC}(\iota) = \mathrm{TC}(\iota)$.

For the particular case $\iota = \mathrm{id}_{S^m}$, $\dim S^m < 2m-1$ for any $m \geqslant 2$, so we have $\mathrm{relTC}(S^m) = \mathrm{TC}(S^m)$ for any $m \geqslant 2$.

3. Open problems

Let be given a map $\iota_X\colon A \to X$. Consider the map $\alpha_i\colon A \to G_i(\iota_X)$ of the Ganea construction 1. In [3], we showed that $\mathrm{relcat}(\alpha_i) = \mathrm{secat}(\alpha_i) = i$ for $i \leqslant \mathrm{secat}(\iota_X)$ and $\mathrm{relcat}(\alpha_i) = \mathrm{relcat}(\iota_X)$ for $i \geqslant \mathrm{relcat}(\iota_X)$. We have no evidence that $\mathrm{relcat}(\alpha_i) = \mathrm{secat}(\alpha_i)$ for any i but we think it would be true:

Conjecture 21. *For any map $\iota_X\colon A \to X$, any $i \geqslant 0$, we have*

$$\mathrm{secat}(\alpha_i) = \mathrm{relcat}(\alpha_i) = \min\{i, \mathrm{relcat}(\iota_X)\}.$$

Another more tricky conjecture is:

Conjecture 22. *For any map $\iota_X\colon A \to X$, if ι_X has a homotopy retraction, then we have* $\mathrm{secat}(\iota_X) = \mathrm{relcat}(\iota_X)$.

A positive answer to this question would imply that $\mathrm{TC}(X) = \mathrm{relTC}(X)$ for *any* X and even $\mathrm{TC}(\iota) = \mathrm{relTC}(\iota)$ for *any* map $\iota_X\colon A \to X$, since $(\mathrm{id}_A, \iota_X)\colon A \to A \times X$ has an obvious (homotopy) retraction $\mathrm{pr}_1\colon A \times X \to A$.

As the referee noticed, it is likely that relTC equals the *monoidal* topological complexity introduced by Iwase and Sakai [5].

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